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# The Transformation Properties at the Quantum Level in Field Theories for a Gauge-Invariant System with a Higher-Order Lagrangian

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**Abstract** Based on the configuration-space generating functional of the Green functions for the gauge-invariant system in higher-order derivatives theories, the equations of the transformation properties at the quantum level have been derived. It follows that the sufficient conditions are found which implies that there exists the conservation laws and the expressions of the quantal conserved laws are also given. Applying the results to the non-Abelian Chern-Simons higher-order derivatives theories, the quantal BRST conserved charge and other conserved charges are found, the transformation properties of the conformal transformation at the quantum level is discussed, the quantal conserved angular momentum is derived, it is pointed out that fractional spin in this system may be also preserved in quantum theories. But the connection between the symmetries and conservation laws in classical theories are not always preserved in quantum theories.

**Keywords** Gauge field · Higher-order derivatives theories · Symmetries and conservation laws · Chern-Simons theories

## 1 Introduction

Symmetry is now a basic concept in modern field theories, the connection between global symmetry and conservation law is usually referred to as first Noether theorem in classical theory. The classical second Noether theorem refers to a local symmetry of a system

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which implies that there exist some differential identities (Noether identities). Noether identity corresponds to the Ward identity in quantum theory. As is well known, classical Noether theorems and Ward identities are usually formulated in terms of Lagrange's variables in configuration space. The dynamical system can also be described in phase space. The canonical formalism for a system with a singular Lagrangian is characterized by the presence of certain constraints among the canonical variables (constrained Hamiltonian system) [1, 2], such as Gauss constraints in electromagnetic field and Yang-Mills field, Virasor conditions in string theories etc. A system with a gauge-invariant (local invariance) Lagrangian is subject to some inherent phase-space constraints, which is a constrained Hamiltonian system.

A system with a gauge-invariant Lagrangian is a constrained Hamiltonian system. Recently the global/local canonical symmetries for a system with a regular/singular Lagrangian in classical/quantum theory have been studied. The classical canonical Noether theorems/Neother identities and Poincaré-Cartan integral invariant are established [3–6], the canonical ward identities [7–9], the quantum canonical Noether theorem [10–15] /Noether identities [16] and Poincaré-Cartan integral invariant [17] in quantum theories are also formulated and some applications are given.

The quantization of those system can be formulated with aid of the Dirac formalism of the constrained Hamiltonian system [1] and the method of path integral in phase space [18, 19]. However, for a gauge-invariant system one can conveniently use the FP (Faddeev-Popov) trick to formulate its path integral quantization in configuration space. The phase-space path integrals are more fundamental than configuration-space path integral. In certain case (for example, Yang-Mills theory), according to the path integral quantization of constrained Hamiltonian system (Senjanovic scheme /Batalin-Fradkin-Vilkovisky scheme), one can carry out explicit integration over momenta in the phase-space path integral which can be converted to the same results obtained by using the FP trick [19], this trick is a simple and more useful method to give the quantization for the gauge fields, thus, so one can conveniently study the quantum symmetry of the gauge system in configuration space.

The global symmetry yields conservation law, all of those considerations in that case are based on the invariance of an action of the system under the global transformation both in classical and quantum case. Now let us consider a more general case in which the action of the gauge-invariant (local invariance) system is variant under the global transformation in configuration space, we shall further study the transformation properties of such a system at the quantum level, the preliminary studies was performed for a system with a finite number of degrees of freedom in the previous works [20]. As is well known, in classical theories, those equations of transformation properties corresponding the spatial translation yields the momentum theorem and spatial rotation yields the moment of momentum theorem etc respectively [21, 22]. Some results hold true in quantum mechanics for a system with a finite number of degrees of freedom [20]. In this paper we discuss the quantal transformation properties of a gauge-invariant system with a higher-order Lagrangian in field theories and give some applications. Dynamical system described in terms of higher-order Lagrangian has close relation with the gravity theory, modified KdV equations, supersymmetry, string model and other problems [4]. The use of higher-order derivative terms in the Lagrangian allows us to improve the behavior of the correspondent propagators at large momentum, rendering the theory less divergent, it has attracted much attention recently.

The paper is organized as follows. In Sect. 2, based on the configuration-space generating functional of the Green functions obtained by using the FP trick for a gauge-invariant system with a higher-order Lagrangian in field theories, consider an action of this system is variant under the global transformation in configuration space, the equations of the transformation properties at the quantum level have been formulated for this gauge-invariant system.

From those quantal equations of the transformation properties, the sufficient condition are obtained to show that there exist some conservation laws at the quantum level and the expressions of those quantal conservation laws are also given for such a system. In Sect. 3, we give some applications of the results to the non-Abelian CS (Chern-Simons) theories with higher-order Lagrangian, the quantal BRST (Becchi-Rouet-Stora-Tyutin) conserved charge is found for this system and another quantal conserved charge is also obtained by using quantal second Noether theorem [16, 23]. The transformation properties of the conformal transformation of this system at the quantum level have been studied, the quantal conserved generalized moment of momentum is derived, in which one needs to take into account the contribution of angular momentum of ghost fields. It is pointed out that the property of fractional spin is preserved at the quantum level in higher-order derivatives CS theories. But the connection between the symmetries and conservation laws in classical theories are not always preserved in quantum theories. Section 4 is devoted to conclusions and discussions.

## 2 The Properties of the Transformation at the Quantum Level

Let us consider a gauge-invariant field described by $n$ field variables $\varphi^\alpha(x)$ ($\alpha = 1, 2, \ldots, n$), $x = (x_0, x_i)$ ($x_0 = t,\ i = 1, 2, 3$) and the motion of the field defined by a Lagrangian involving higher-order derivatives in the form of a functional

$$L = L[\varphi^\alpha_{(0)}, \varphi^\alpha_{(1)}, \ldots, \varphi^\alpha_{(N)}] = \int_V \mathcal{L}(\varphi^\alpha \ldots, \varphi^\alpha_{,\mu}, \ldots, \varphi^\alpha_{,\mu(N)}) d^3x \tag{1a}$$

where

$$\begin{aligned} &\varphi^\alpha_{(0)} = \varphi^\alpha, \qquad \varphi^\alpha_{(1)} = D\varphi^\alpha = \frac{d}{dt}\varphi^\alpha = \dot{\varphi}^\alpha, \qquad \ldots, \\ &\varphi^\alpha_{,\mu} = \partial_\mu \varphi^\alpha = \frac{\partial}{\partial x^\mu}\varphi^\alpha, \qquad \ldots, \qquad \varphi^\alpha_{,\mu(m)} = \underbrace{\partial_\mu \cdots \partial_\nu}_{m} \varphi^\alpha. \end{aligned} \tag{1b}$$

For this gauge system the configuration–space generating functional of the Green function can be found by using the FP trick through a transformation of path integral [24]

$$Z[J] = \int \mathcal{D}\phi^a \exp\left\{ i \int d^4x (\mathcal{L}_{\text{eff}} + J_a \phi^a) \right\} \tag{2a}$$

where

$$\mathcal{L}_{\text{eff}} = \mathcal{L} + \mathcal{L}_{\text{f}} + \mathcal{L}_{\text{gh}} \tag{2b}$$

and $\mathcal{L}$ is a gauge-invariant Lagrangian density, $\mathcal{L}_{\text{f}}$ is determined by the gauge conditions and $\mathcal{L}_{\text{gh}}$ is a ghost term, $\phi^a$ represent all field variables (including the ghost fields) and $J_a$ represent the exterior sources.

Now we study the quantal transformation properties of the system under the global transformation. It is supposed that the variation of the effective action $I_{\text{eff}} = \int d^4x \mathcal{L}_{\text{eff}}$ under the global transformation

$$\begin{cases} x^{\mu'} = x^\mu + \delta x^\mu = x^\mu + \varepsilon_\sigma \tau^{\mu\sigma}(x, \ldots, \phi^a_{,\mu(m)}, \ldots), \\ \phi^{a'}(x') = \phi^a(x) + \Delta' \phi^a(x) = \phi^a(x) + \varepsilon_\sigma \xi^{a\sigma}(x, \ldots, \phi^a_{,\mu(m)}, \ldots) \end{cases} \tag{3}$$

is given by

$$\delta I_{\text{eff}} = \varepsilon_\sigma \int d^4x[\partial_\mu W^{\mu\sigma}(x,\phi^a,\ldots,\phi^a_{,\mu(m)},\ldots) + R^a(x,\phi^a,\ldots,\phi^a_{,\mu(m)},\ldots)] \tag{4}$$

where $\varepsilon_\sigma$ ($\sigma = 1,\ldots,r$) are infinitesimal arbitrary parameters, $\tau^{\mu\sigma}$, $\xi^{a\sigma}$, $W^{\mu\sigma}$ and $R^\sigma$ are some functions. Now we localized the global transformation (3) and consider the following local transformation connected with the global transformation (3)

$$\begin{cases} x^{\mu'} = x^\mu + \Delta x^\mu = x^\mu + \varepsilon_\sigma(x)\tau^{\mu\sigma}(x,\ldots,\phi^a_{,\mu(m)},\ldots), \\ \phi^{a'}(x') = \phi^a(x) + \Delta\phi^a(x) = \phi^a(x) + \varepsilon_\sigma(x)\xi^{a\sigma}(x,\ldots,\phi^a_{,\mu(m)},\ldots) \end{cases} \tag{5}$$

where $\varepsilon_\sigma(x)$ are infinitesimal arbitrary functions and their values and derivatives vanish on the boundary of the space-time domain. Under the transformation (5) the variation of the effective action is given by [19]

$$\begin{aligned} \Delta I_{\text{eff}} = \int d^4x\varepsilon_\sigma(x)\Bigg\{ & \frac{\delta I_{\text{eff}}}{\delta\phi^a}(\xi^{a\sigma} - \phi^a_{,\mu}\tau^{\mu\sigma}) \\ & + \partial_\mu\Bigg[\mathcal{L}_{\text{eff}}\tau^{\mu\sigma} + \sum_{n=0}^{N-1}\prod_a^{\mu\nu(n)}\partial_{\nu(n)}(\xi^{a\sigma} - \phi^a_{,\rho}\tau^{\rho\sigma})\Bigg]\Bigg\} \\ & + \int d^4x\Bigg\{\Bigg[\mathcal{L}_{\text{eff}}\tau^{\mu\sigma} + \sum_{n=0}^{N-1}\prod_a^{\mu\nu(n)}\partial_{\nu(n)}(\xi^{a\sigma} - \phi^a_{,\rho}\tau^{\rho\sigma})\Bigg]\partial_\mu\varepsilon_\sigma(x)\Bigg\} \end{aligned} \tag{6}$$

where

$$\frac{\delta I_{\text{eff}}}{\delta\phi^a} = \sum_{m=0}^{N}(-1)^m\partial_{\mu(m)}\mathcal{L}^{\mu(m)}_{a\text{eff}}, \tag{7a}$$

$$\mathcal{L}^{\mu(m)}_{a\text{eff}} = \frac{1}{m!}\sum_{\substack{\text{all permutation} \\ \text{of indices } \mu(m)}}\frac{\partial\mathcal{L}_{\text{eff}}}{\partial\phi^a_{,\mu(m)}}, \tag{7b}$$

$$\prod_a^{\mu\nu(m)} = \sum_{l=0}^{N-(m+1)}(-1)^l\partial_{\lambda(l)}\mathcal{L}^{\mu\nu(m)\lambda(l)}_{a\text{eff}}. \tag{7c}$$

Since the variation of an effective action under the global transformation (3) is given by (4). Substituting (4) into (6) and in according with the boundary conditions of $\varepsilon_\sigma(x)$, then the expression (6) can be written as

$$\Delta I^p_{\text{eff}} = \int d^4x\varepsilon_\sigma(x)\Bigg\{\partial_\mu\Bigg[W^{\mu\sigma} - \mathcal{L}_{\text{eff}}\tau^{\mu\sigma} - \sum_{n=0}^{N-1}\prod_a^{\mu\nu(n)}\partial_{\nu(n)}(\xi^{a\sigma} - \phi^a_{,\rho}\tau^{\rho\sigma})\Bigg] + R^\sigma\Bigg\}. \tag{8}$$

Let us suppose that the Jacobian of the transformation (5) is $\bar{J}[\phi^a,\ldots,\phi^a_{,\mu(m)},\ldots,\varepsilon^\sigma]$, the invariance of the generating functional (2a) under the transformation (5) implies that $\delta Z/\delta\varepsilon_\sigma(x) = 0$. Substituting (5) and (8) into (2a) and functionally differentiating it with respect to $\varepsilon_\sigma(x)$, one obtains

$$\int \mathcal{D}\phi^a \left\{ \partial_\mu \left[ W^{\mu\sigma} - \mathcal{L}_{\text{eff}}\tau^{\mu\sigma} - \sum_{n=0}^{N-1} \prod_a^{\mu\nu(n)} \partial_{\nu(n)}(\xi^{a\sigma} - \phi^a_{,\rho}\tau^{\rho\sigma}) \right] + R^\sigma + J_0^\sigma + M^\sigma \right\}$$
$$\times \exp\left\{ iI_{\text{eff}} + i\int d^4x J_a\phi^a \right\} = 0 \tag{9}$$

where

$$J_0^\sigma = -i\delta\bar{J}/\delta\varepsilon_\sigma(x)|_{\varepsilon_\sigma(x)=0}, \tag{10a}$$
$$M^\sigma = J_a(\xi^{a\sigma} - \phi^a_{,\mu}\tau^{\mu\sigma}). \tag{10b}$$

Functionally differentiating (9) with respect to $J_a(x)$ a total of $n$ times, one obtains

$$\int \mathcal{D}\phi^a \Bigg( \left\{ \partial_\mu \left[ W^{\mu\sigma} - \mathcal{L}_{\text{eff}}\tau^{\mu\sigma} - \sum_{n=0}^{N-1} \prod_a^{\mu\nu(n)} \partial_{\nu(n)}(\xi^{a\sigma} - \phi^a_{,\rho}\tau^{\rho\sigma}) \right] + R^\sigma + J_0^\sigma + M^\sigma \right\}$$
$$\times \phi^e(x_1)\cdots\phi^f(x_n) - i\sum_j \phi^e(x_1)\cdots\phi^c(x_{j-1})\phi^d(x_{j+1})\cdots\phi^f(x_n)N^{a\sigma}\delta(x-x_j) \Bigg)$$
$$\times \exp\left\{ iI_{\text{eff}} + i\int d^4x J_a\phi^a \right\} = 0 \tag{11}$$

where

$$N^{a\sigma} = \xi^{a\sigma} - \phi^a_{,\mu}\tau^{\mu\sigma}. \tag{11a}$$

Let $J_a = 0$ in (11), one gets

$$\langle 0|T^* \Bigg( \left\{ \partial_\mu \left[ W^{\mu\sigma} - \mathcal{L}_{\text{eff}}\tau^{\mu\sigma} - \sum_{n=0}^{N-1} \prod_a^{\mu\nu(n)} \partial_{\nu(n)}(\xi^{a\sigma} - \phi^a_{,\rho}\tau^{\rho\sigma}) \right] + R^\sigma + J_0^\sigma \right\}$$
$$\times \phi^e(x_1)\cdots\phi^f(x_n)$$
$$- i\sum_j \phi^e(x_1)\cdots\phi^c(x_{j-1})\phi^d(x_{j+1})\cdots\phi^f(x_n)N^{a\sigma}\delta(x-x_j) \Bigg)|0\rangle = 0 \tag{12}$$

where $T^*$ stands for the covariantized $T$ product [25]. Fixing $t$ and letting

$$t_1, t_2, \ldots, t_k \to +\infty, \qquad t_{k+1}, t_{k+2}, \ldots, t_n \to -\infty$$

and using the reduction formula [25], one can write expression (12) as

$$\langle \text{out}, k| \left\{ \partial_\mu \left[ W^{\mu\sigma} - \mathcal{L}_{\text{eff}}\tau^{\mu\sigma} - \sum_{n=0}^{N-1} \prod_a^{\mu\nu(n)} \partial_{\nu(n)}(\xi^{a\sigma} - \phi^a_{,\rho}\tau^{\rho\sigma}) \right] + R^\sigma + J_0^\sigma \right\}$$
$$\times |n-k, \text{in}\rangle = 0. \tag{13}$$

Since $k$, $n$ are arbitrary, one obtains

$$\partial_\mu j^{\mu\sigma} + R^\sigma + J_0^\sigma = 0, \tag{14a}$$

$$j^{\mu\sigma}=W^{\mu\sigma}-\mathcal{L}_{\text{eff}}\tau^{\mu\sigma}-\sum_{n=0}^{N-1}\prod_{a}^{\mu\nu(n)}\partial_{\nu(n)}(\xi^{a\sigma}-\phi^{a}_{,\rho}\tau^{\rho\sigma}). \tag{14b}$$

Thus, we obtain the general equations (14) of transformation properties at the quantum level for a gauge-invariant higher-order derivatives system with a non-invariance effective action and a change of integral measure under the global transformation (3) in configuration space. We take the integral of (14a) on the 3-dimensional space domain, if we assume that the fields have a configuration, which vanishes rapidly at spatial infinity, then using the Gauss theory, we obtain

$$D\int_V d^3x\left[\mathcal{L}_{\text{eff}}\tau^{0\sigma}+\sum_{n=0}^{N-1}\prod_{a}^{0\nu(n)}\partial_{\nu(n)}(\xi^{a\sigma}-\phi^{a}_{,\rho}\tau^{\rho\sigma})-W^{0\sigma}\right]=\int_V d^3x(R^\sigma+J_0^\sigma). \tag{15}$$

Consequently, we obtain the following theorem: If an effective Lagrangian (2b) is invariant up to a 4-dimensional divergence term under the global transformation (3), i.e. $R^\sigma=0$ in expression (14a), and the Jacobian of the corresponding transformation (5) is independent of $\varepsilon_\sigma(x)$, thus, $J_0^\sigma=0$, then there are some conserved quantities at the quantum level for a gauge-invariant system with a higher-order Lagrangian in field theories.

$$Q^\sigma=\int_V d^3x\left[\mathcal{L}_{\text{eff}}\tau^{0\sigma}+\sum_{n=0}^{N-1}\prod_{a}^{0\nu(n)}\partial_{\nu(n)}(\xi^{a\sigma}-\phi^{a}_{,\rho}\tau^{\rho\sigma})-W^{0\sigma}\right]$$
$$(\sigma=1,2,\ldots,r). \tag{16}$$

This results can be called quantal first Noether theorem for a gauge-invariant system with a higher-order Lagrangian. It shows that the classical relation of global symmetries to the conservation laws are not always hold true in quantum theories if the integral measure is variant ($J_0^\sigma\neq 0$) under the global transformation (3). When we take the change of integral measure into account, for a system which effective action is not invariant under the global transformation (3), there may also have some conserved quantities at the quantum level if the conditions $R^\sigma+J_0^\sigma=0$ are satisfied, but, in general, there are no corresponding classical conserved quantities in this case.

## 3 Higher-Derivatives Non-Abelian CS Theories

Numerous recent investigations of (2 + 1)-dimensional gauge theories with CS (Chern-Simons) terms in the Lagrangian have revealed the occurrence of fractional spin and statistics [11, 26]. It has important sense to explain the quantum Hall effect and high-$T_C$ superconductivity. Theoretical understanding of them has been gained in context of both quantum mechanics and field theory. So far, development in the direction of field theory has not progressed as for as that of the quantum mechanics. A few work was investigated for 1st order–derivatives non-Abelian CS theories [26]. Now we shall further study the quantal symmetries and fractional spin in higher-derivatives non-Abelian CS theories.

Let us consider the (2 + 1)-dimensional non-Abelian CS gauge field $A^a_\mu$ coupled to the matter field $\psi$, whose Lagrangian is given by [27]

$$\mathcal{L}=-\frac{c^2}{4\pi}D_\rho F^a_{\mu\nu}D^\rho F^{a\mu\nu}-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}+\frac{\kappa}{4\pi}\varepsilon^{\mu\nu\rho}\left(\partial_\mu A^a_\nu A^a_\rho+\frac{1}{3}f^{abc}A^a_\mu A^b_\nu A^c_\rho\right)$$
$$+i\bar{\psi}\gamma^\mu D_\mu\psi \tag{17a}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^a_{bc} A^b_\mu A^c_\nu \tag{17b}$$

and $D_\mu$ stands for covariant derivatives, and $f^a_{bc}$ are structure constants of the gauge group. The gauge invariance of non-Abelian CS term requires the quantization of the dimensionless constant $\kappa$, $\kappa = \frac{n}{4\pi}$ ($n \in Z$). Dirac $\boldsymbol{\gamma}$-matrices are $\gamma^0 = \sigma^3$, $\gamma^1 = i\sigma^1$, and $\gamma^2 = i\sigma^2$ ($\sigma'$s are the Pauli matrices). We adopt the FP trick to give the quantization of this system in configuration space. Let us consider following gauge conditions

$$\partial^\mu A^a_\mu = \lambda^a \tag{18}$$

where the functions $\lambda^a$ are independent of $A^a_\mu(x)$, using the FP trick are can formulate the generating functional for this system in configuration space as [19]

$$Z[J, \bar{\eta}, \eta] = \int \mathcal{D}A^a_\mu \mathcal{D}\bar{\psi}\mathcal{D}\psi \det M_L \delta(\partial^\mu A^a_\mu - \lambda^a) \times \exp\left[i \int d^3x (\mathcal{L} + J^\mu_a A^a_\mu + \bar{\psi}\eta + \bar{\eta}\psi)\right] \tag{19a}$$

where $M_L = [M^{ab}_L]$, and

$$M^{ab}_L = (\delta^{ab}\partial_\mu\partial^\mu - f^a_{bc} A^c_\mu \partial^\mu)\delta^4(x-y). \tag{19b}$$

According to the integral properties of the Grassmann variables $\bar{C}_a(x)$ and $C_b(y)$, one has

$$\det M_L = \int \mathcal{D}\bar{C}_a(x)\mathcal{D}C_b(y) \exp\left[i \int d^3x d^3y \bar{C}_a(x) M^{ab}_L C_b(y)\right]. \tag{20}$$

Multiplying (19a) by the following factor

$$\exp\left\{i \int d^3x \left[\lambda_a B^a + \frac{\alpha_0}{2}(B^a)^2\right]\right\}$$

where $B^a$ are auxiliary scalar fields, $\alpha_0$ is a parameter, and then taking the path integral of the result with respect to $\lambda^a$ and $B^a$, one gets

$$Z[J, \bar{\eta}, \eta, \bar{\xi}, \xi] = \int \mathcal{D}A^a_\mu \mathcal{D}\bar{\psi}\mathcal{D}\psi \mathcal{D}\bar{C}_a \mathcal{D}C_a \mathcal{D}B^a \times \exp\left\{i \int d^3x [\mathcal{L}_{\text{eff}} + J^\mu_a A^a_\mu + \bar{\psi}\eta + \bar{\eta}\psi + \bar{\xi}^a C_a + \bar{C}_a \xi^a]\right\} \tag{21}$$

where $\mathcal{L}_{\text{eff}}$ is an effective Lagrangian density

$$\mathcal{L}_{\text{eff}} = \mathcal{L} + B^a \partial^\mu A^a_\mu + \frac{\alpha_0}{2}(B^a)^2 - \partial^\mu \bar{C}_a D^a_{b\mu} C_b, \tag{22a}$$

$$D^a_{b\mu} = \delta^a_b \partial_\mu + f^a_{cb} A^c_\mu \tag{22b}$$

where $\mathcal{L}$ is Lagrangian density (17a), $\xi^a$ and $\bar{\xi}^a$ are the exterior sources with respect to ghost field $\bar{C}_a$ and $C_a$ respectively.

Obviously, the expression (22a) is invariant under the following transformation,

$$C^a \to e^{\theta} C^a, \qquad \bar{C}^a \to e^{-\theta} \bar{C}^a \tag{23}$$

where $\theta$ is a parameter, the Jacobian of the transformation (23) is equal to unity. Using the results (16) of quantal first Noether theorem, one obtains the conserved quantity at the quantum level

$$Q_c = \int d^2x J_c^0, \tag{24a}$$

$$J_c^{\mu} = \bar{C}^a D_b^{a\mu} C^b, \tag{24b}$$

$Q_c$ are the number of ghosts which implies that the numbers of ghosts are conserved in quantum theories.

### 3.1 BRST symmetry

Under the following BRST transformation

$$\begin{cases} \delta A_\mu^a = -\tau D_{b\mu}^a C^b, & \delta B^a = 0, \\ \delta\psi = i\tau T^a C^a \psi, & \delta\bar{\psi} = -i\tau\bar{\psi} T^a C^a, \\ \delta C^a = \frac{1}{2}\tau f_{bc}^a C^b C^c, & \delta\bar{C}^a = \tau B^a \end{cases} \tag{25}$$

the change of the effective Lagrangian density is given by

$$\delta\mathcal{L}_{\text{eff}} = -\tau\partial^{\mu} G_{\mu}, \qquad G_{\mu} = B^a D_{b\mu}^a C^b \tag{26}$$

where $T^a$ are the generators of gauge group, $\tau$ is a Grassmann parameter, the Jacobian of the BRST transformation (25) is equal to unity, in this case the quantal conserved currents $j^{\nu}(\partial_{\nu} j^{\nu} = 0)$ are given by

$$\begin{aligned} j^{\nu} = {} & \left[\frac{\partial\mathcal{L}_{\text{eff}}}{\partial A_{\mu,\nu}} - \partial_{\rho}\left(\frac{\partial\mathcal{L}_{\text{eff}}}{\partial A_{\mu,\nu\rho}^a}\right)\right] D_{b\mu}^a C^b + \frac{\partial\mathcal{L}_{\text{eff}}}{\partial A_{\mu,\nu\rho}^a}\partial_{\rho}(D_{b\mu}^a C^b) + \frac{\partial\mathcal{L}_{\text{eff}}}{\partial C_{,\nu}^a} + \frac{\partial\mathcal{L}_{\text{eff}}}{\partial\bar{C}_{,\nu}^a} \\ & + B^a D_b^{a\nu} C^b + \frac{\partial\mathcal{L}_{\text{eff}}}{\partial\psi_{,\nu}}\delta\psi + \frac{\partial\mathcal{L}_{\text{eff}}}{\partial\bar{\psi}_{,\nu}}\delta\bar{\psi} \end{aligned} \tag{27}$$

and the quantal BRST conserved quantity is given by

$$Q = \int d^2x (P_a^{\mu}\delta A_\mu^a + Q_a^{\mu}\delta B_\mu^a + \bar{\pi}\delta\psi + \delta\bar{\psi}\pi + \bar{R}_a\delta C^a + \delta\bar{C}^a R_a + B^a D_b^{a0} C^b) \tag{28}$$

where

$$B_\mu^a = \dot{A}_\mu^a, \tag{29a}$$

$$\begin{aligned} P^{a\mu} & = \frac{\partial\mathcal{L}_{\text{eff}}}{\partial\dot{A}_\mu} - 2\partial_k\left(\frac{\partial\mathcal{L}_{\text{eff}}}{\partial\dot{A}_{\mu,k}^a}\right) - \partial_0\frac{\partial\mathcal{L}_{\text{eff}}}{\partial\ddot{A}_\mu^a} \\ & = F^{a\mu 0} + \frac{\kappa}{4\pi}\varepsilon^{0\mu\nu} A_\nu^a - \frac{c^2}{4\pi} D_i D^i F^{a\mu 0} - D_0 Q^{a\mu} - \frac{c^2}{\pi} D_i D_0 F^{a\mu i}, \end{aligned} \tag{29b}$$

$$Q_a^{\mu} = \frac{\partial \mathcal{L}_{\text{eff}}}{\partial \ddot{A}_{\mu}^{a}} = \frac{c^2}{\pi} D_0 F_a^{\mu 0}, \tag{30}$$

$$\begin{aligned} &\bar{\pi} = -i\bar{\psi}\gamma^0, \qquad \pi = 0, \\ &R_a = -\dot{\bar{C}}^a, \qquad \bar{R}_a = D_b^{a0} C^b. \end{aligned} \tag{31}$$

This method to derived the quantal conserved quantity here is more convenient than that of the phase-space path integral of constrained Hamiltonian system [28].

If we only consider the change of the fields $A_\mu^a$, $\psi$ and $\bar{\psi}$, fixing the ghost fields $C^a$, $\bar{C}^a$ and auxiliary fields $B^a$ in the BRST transformation, we have

$$\begin{cases} \delta A_\mu^a = -\tau D_{b\mu}^a C^b, \\ \delta\psi = i\tau T^a C^a \psi, \qquad \delta\bar{\psi} = -i\tau\bar{\psi} T^a C^a, \\ \delta C^a = \delta\bar{C}^a = \delta B^a = 0. \end{cases} \tag{32}$$

Under the transformation (32), the change of an effective Lagrangian density is given by

$$\delta\mathcal{L}_{\text{eff}} = U_a \varepsilon^a(x) + f_{be}^a(\partial^\mu \bar{C}^a C^e - B^a A^{e\mu})\partial_\mu \varepsilon^b) + B^b \partial^2 \varepsilon^b] \tag{33}$$

where $\varepsilon^a(x) = \tau C^a(x)$, and $U_a(\theta)$ do not contain the derivatives of the $\varepsilon^a(x)$. In this case $R^a \neq 0$ in (14), one can not be derived the quantal conserved quantities by using the results (16) of the quantal first Noether theorem but one can use quantal second Noether theorem [16] to derive another quantal conserved quantities (see (4.8) in Ref. [16])

$$Q_P = \int d^2x[P_a^\mu \delta A_\mu^a + Q_a^\mu \delta B_\mu^a + \bar{\pi}\delta\psi + \delta\bar{\psi}\pi + f_{be}^a(\dot{\bar{C}}^a C^e C^b - B^a A^{e0} C^b) + B^a C^a]. \tag{34}$$

### 3.2 Conformal Symmetry

The special conformal group contains a fifteen parameters Lie group, the Poincaré group is its subgroup, it also include a space-time dilation and space-time inversions. The effective Lagrangian density (22) is invariant under the translation group, the Jacobian of the transformation of the field variables is equal to unity, according to expressions (16) one can obtain the conserved generalized energy-momentum at the quantum level

$$H = \int d^2x\{P_a^\mu \dot{A}_\mu^a + Q_a^\mu \ddot{A}_\mu^a + \bar{\pi}\dot{\bar{\psi}} + \pi\dot{\psi} + R_a \dot{C}^a + \bar{R}_a \dot{\bar{C}}^a - \mathcal{L}_{\text{eff}}\}, \tag{35}$$

$$P_i = \int d^2x\{P_a^\mu \partial_i A_\mu^a + Q_a^\mu \partial_i \dot{A}_\mu^a + \bar{\pi}\partial_i\bar{\psi} + \pi\partial_i\psi + R_a\partial_i C^a + \bar{R}_a \partial_i \bar{C}^a\}. \tag{36}$$

Under the rotation in $(x_1, x_2)$ plan, the Jacobian of the transformation of the field variables is also equal to unity, this invariance of the effective Lagrangian density (22) implies there exist the conserved generalized angular momentum at the quantum level

$$J_{12} = \int d^2x\left\{P_a^\mu\left(x_2\frac{\partial A_\mu^a}{\partial x_1} - x_1\frac{\partial A_\mu^a}{\partial x_2}\right) + Q_a^\mu\left(x_2\frac{\partial B_\mu^a}{\partial x_1} - x_1\frac{\partial B_\mu^a}{\partial x_2}\right) + P_a^\mu\left(\left(\sum_{12}\right)_{\mu\nu}\right)A_a^\nu\right.$$
$$+\bar{\pi}_a\left(x_2\frac{\partial\psi^a}{\partial x_1} - x_1\frac{\partial\psi^a}{\partial x_2}\right) + \left(x_2\frac{\partial\bar{\psi}^a}{\partial x_1} - x_1\frac{\partial\bar{\psi}^a}{\partial x_2}\right)\pi_a + \bar{\pi} S_{12}\psi$$

$$+\bar{R}_a\left(x_2\frac{\partial C^a}{\partial x_1}-x_1\frac{\partial C^a}{\partial x_2}\right)+\left(x_2\frac{\partial \bar{C}^a}{\partial x_1}-x_1\frac{\partial \bar{C}^a}{\partial x_2}\right)R_a\Bigg\} \tag{37a}$$

where

$$\left(\sum_{\rho\sigma}\right)_{\mu\nu}=g_{\rho\mu}g_{\sigma\nu}-g_{\rho\nu}g_{\sigma\mu}, \tag{37b}$$

$$S_{\mu\nu}=\frac{1}{4}[\gamma_\mu,\gamma_\nu]. \tag{37c}$$

Thus, we see that in non-Abelian higher-derivatives CS theory the quantal conserved generalized energy-momentum and angular momentum differ from the results derived from classical Noether theorem in that one needs to take into account the corresponding contributions of ghost fields.

The more inspired feature of the CS theory is that anyonic quantum field theory which have revealed the occurrence of fractional spin. In the study of anyons at the field-theoretical level, the Abelian CS theory minimally coupled to the matter fields is usually considered as the base system, numerous work has been given to investigate for those systems [2, 11]. Development in the direction of the non-Abelian CS theory [26] has not progress as far as that the Abelian CS theory, and all the considerations are based the Lagrangian of the system only involving first order derivatives. For non-Abelian higher-derivatives CS theory, it has been pointed that one needs to take into account the angular momentum contribution of the ghost fields in (37a), but the CS coefficient $\kappa$ does not appear both in the terms of the ghost fields and higher-derivatives terms, thus the property of the fractional spin described by using (37a) is no influence [26]. Consequently, the fractional spin and statistics in non-Abelian CS theory with a higher-order Lagrangian is also preserved in quantum theories.

It is to check that the effective Lagrangian density (22) is invariant under the Poincaré group, but it does not has the invariance of the space-time dilation transformation and the space-time inversion transformation, so the effective Lagrangian density (22) is not invariant under the conformal group. Although the effective Lagrangian density (22) is not invariant under the space-time dilation and inversion transformation, there is no corresponding conservation laws at the quantum level, but from (14) one can also derive the quantal equations of transformation properties. The connection between the symmetries and conservation laws in classical theories are not always preserved in quantum theories.

## 4 Conclusions and Discussions

The path-integral quantization for the higher-derivatives gauge system is formulated by using the FP trick in configuration space and an effective Lagrangian in configuration-apace generating function of the Green function is obtained. Consider the change of an effective Lagrangian under the global transformation, based on those generating functional, the quantal equations of the transformation propertied for the higher-derivatives gauge system have been derived. It follows that the sufficient conditions are given to show that there are some conservation laws and the expression of the conservation laws for such a system at the quantum level are also obtained (quantal first Noether theorem). The application of above results to the higher-derivatives non-Abelian CS theory, the quantal conserved BRST charge is derived by using quantal first Noether theorem and another quantal conserved charge is also obtained by using quantal second Noether theorem. The quantal properties of the conformal transformation for such a system have been studied, the quantal conserved angular momentum is deduced, although one needs to take into account the angular momentum contribution of the ghost fields, the fractional spin in this system is also preserved. The effective Lagrangian of this system is not invariant under the conformal transformation, the connection between the symmetries and conservation laws in classical theories are not preserved in quantum theories for this system.

In this paper we have studied the transformation properties at the quantum level for a gauge-invariant system with a higher-order Lagrangian in configuration space. The path integral provide a useful tool, the advantage of the path integral scheme to study the quantal transformation properties is that we dealt with C-number, not q-number. The phase-space path integrals are more fundamental than configuration-space path integrals, in certain case the configuration-space path integral can be obtained by carrying out explicit integration over the canonical momenta in the phase-space path integral. For a gauge-invariant system a simpler and more useful method to give its quantization is the FP trick. But for a system if the FP trick is useless, one must adopt phase-space path integral to quantize such a system.

The CS theory has close relation with the anyonic quantum field theory, the study of anyons at the Abelian CS gauge field has some progress both in classical [29] and quantal [2, 11, 26] level. The possibility of the occurrence of fractional spin properties for non-Abelian CS theories was studied mostly at the classical level, whether the conclusion in classical theory is valid at the quantum level, one needs further to study. The problem of fraction spin in the higher-derivatives non-Abelian CS theories will be also discussed in detail. The effective Lagrangian of above higher-derivatives non-Abelian CS model is variant under the conformal transformation, the research of the quantal equations of the such transformation properties arising from expressions (14) and to give their applications, one can still proceed in the same way to study those equations at the quantum level.

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